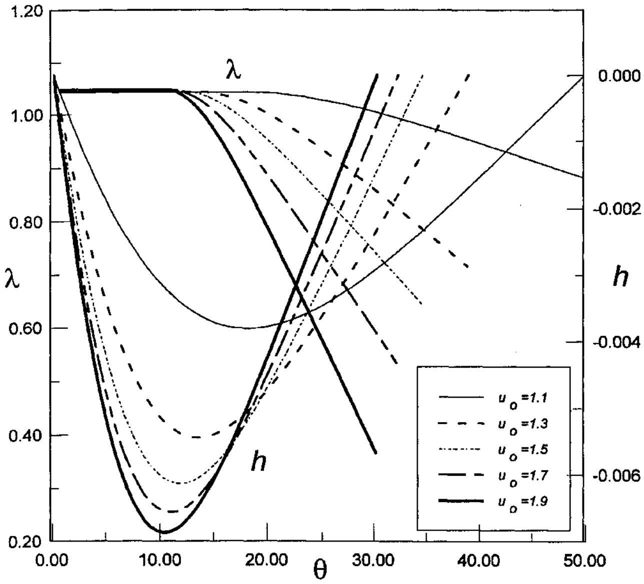

Fig. 2 Trajectories and associated lift controls.

In Eq. (20), λ_0 is the initial normalized lift coefficient from the optimal solution, and k is a constant to be adjusted to meet the final conditions (u_f = given, and $h_f = 0$). For the case of entry speed $u_0 = 1.9$ and exit speed $u_f = 1.0$, with the guidance scheme, using the value of λ_0 from the preceding section, the value of k is then adjusted to 99 to meet the final condition. Using this explicit guidance, the plane change is 27.159 deg, compared to the optimal value of 27.18 deg. Figure 1 shows comparisons of the lift controls and the trajectories for the optimal solution and the explicit guidance solution. The trajectories are almost the same, whereas the controls are slightly different. Several cases of different initial velocities are then studied. Figure 2 is a plot of the lift coefficients and the trajectories. Compared to the optimal solution of $\sigma = 90$ deg, the values of plane change are very good.

Conclusions

A simplified procedure for obtaining a suboptimal solution to the re-entry aeroassisted plane change is invented and an implementable guidance scheme for the lift control of the aeroassisted orbital plane change is proposed. The procedure reduces the number of variables to be guessed from three to two (the initial normalized lift coefficient and the proportional constant). The lifting vehicle enters the atmosphere with the optimal entry angle. The bank angle is maintained at 90 deg, and the lift control follows the simple rule during the atmospheric fly-through phase. The guidance scheme is quite simple, and the resulting trajectories are close to the optimal solutions. The plane change gained is also close to the optimal value.

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Determination of Equilibrium Points for an Aircraft Dynamic Model

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Nomenclature

A	$= b/c$
b, c	= wing span, m, and mean aerodynamic chord, m
C_T	= thrust coefficient, $T/0.5\,\rho V^2 S$
C_W	= weight coefficient, mg/0.5 $\rho V^2 S$
C_l, C_m, C_n	= aerodynamic-moment coefficients in body axes
C_x, C_y, C_z	= aerodynamic-force coefficients in body axes
g	= acceleration of gravity, m/s^2
H	= engine angular momentum, kg m^2/s
h	$= 4H/\rho S b^2 V_e$
I_x, I_y, I_z, I_{xz}	= moments and product of inertia; I_x, I_z, I_{xz} are dimensionless with respect to $\rho S(b/2)^3$ and I_y with respect to $\rho S(c/2)^3$
M	= Mach number
m	= mass, kg
p, q, r	= angular velocity components in body axes; q is dimensionless with respect to $2V_e/c$, p and r with respect to $2V_e/b$
S	= wing surface, m^2
T	= thrust, N
V	= flight speed, m/s
V_e	= equilibrium and reference velocity, m/s
v	= flight speed, dimensionless with respect to V_e
α, β	= aerodynamic angles
$\delta_a, \delta_e, \delta_r, \delta_l$	= aileron, elevator, rudder, and leading-edge flap angles, respectively
μ	= mass density, $2m/\rho Sc$
ρ	= air density, kg/m^3
ϕ, ϑ	= roll and pitch angles, respectively
ω	= angular velocity vector $(p, q, r)^T$

Introduction

A TECHNIQUE for the determination of the equilibrium points of the aircraft equations of motion is presented. The goal is, in particular, a method for the evaluation of the starting points for the continuation algorithms that are used in bifurcation analyses of the nonlinear dynamics of high-performance aircraft to trace the steady states of the system as certain parameters, called continuation parameters, are varied. In the application of bifurcation theory, the failure to determine as many starting points as possible, for an initial choice of the set of control variables, leads to the neglect of certain equilibrium branches, so that some behaviors of the dynamic system may not be revealed. This problem is far from being trivial because the number of equilibrium points is not known a priori and their determination involves the solution of a set of nonlinear, coupled algebraic equations.

In this respect, several theoretical and numerical techniques have been developed, stemming from semianalytical methods[1] to optimization algorithms.[2] In the latter approach a parameter search

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problem is solved, where the number of parameters depends on the restrictions imposed on the equations of motions and/or the aerodynamic model. More recent analyses still rely on this kind of technique when for example the starting points for the bifurcation analysis of a fighter aircraft are first guessed as the equilibrium points of a different, simpler aircraft model.[3] A numerical continuation algorithm[4] is then run for the actual vehicle, so that, if a converged solution is obtained, a steady state has been determined.

It is apparent that a more systematic approach capable of limiting the test-and-trial character of these techniques and/or gaining some physical insight into the map of the aircraft steady states would be of interest.

The equilibria of a reduced-order aircraft model, where gravity is neglected,[3] are determined by a graphical approach. Then, using gravity as the continuation parameter, the equilibrium points of the complete model are calculated for assigned values of the control variables. We consider the bifurcation analysis as the principal field of application of this technique, because it relies on a continuation algorithm when the complete dynamic model is dealt with. However, the proposed method is also of interest when aircraft dynamics are to be investigated in which the zero-gravity assumption has proved meaningful, as in roll-coupling phenomena. We stress that, in this case, all steady states of the vehicle are found by a very simple procedure, which, for a given control vector, provides immediate graphic information on the locations of the equilibrium points in the two-dimensional space of the aerodynamic angles. The proposed technique is applied to the high angle-of-attack model of the F-16, which is reported in detail elsewhere.[5]

Analysis

When the equation for the yaw angle is omitted, because this variable is decoupled from the force and moment equations, the resulting 10th-order set of nondimensional equations of motion is written as follows, in body axes:

$$\begin{aligned}\dot v = {} & (v^2/2\mu)[C_x\cos\alpha\cos\beta + C_y\sin\beta + C_z\sin\alpha\cos\beta \\ & + C_T\cos\alpha\cos\beta + C_W(\sin\beta\cos\vartheta\sin\phi \\ & - \cos\beta\cos\alpha\sin\vartheta + \sin\alpha\cos\beta\cos\vartheta\cos\phi)]\end{aligned} \tag{1}$$

$$\begin{aligned}\dot\alpha = {} & q - \frac{1}{A}(r\sin\alpha + p\cos\alpha)\tan\beta + \frac{v}{2\mu\cos\beta}[C_z\cos\alpha \\ & - C_x\sin\alpha + C_W(\sin\alpha\sin\vartheta + \cos\alpha\cos\vartheta\cos\phi) - C_T\sin\alpha]\end{aligned} \tag{2}$$

$$\begin{aligned}\dot\beta = {} & (1/A)(p\sin\alpha - r\cos\alpha) + (v/2\mu)[-C_x\cos\alpha\sin\beta \\ & + C_y\cos\beta - C_z\sin\alpha\sin\beta + C_W(\cos\alpha\sin\beta\sin\vartheta \\ & + \cos\beta\cos\vartheta\sin\phi - \sin\alpha\sin\beta\cos\vartheta\cos\phi) \\ & - C_T\cos\alpha\sin\beta]\end{aligned} \tag{3}$$

$$\dot p = \frac{\left(I_y/A^3 - I_z\right)qr + I_{xz}(\dot r + pq) + v^2 C_l/A}{I_x} \tag{4}$$

$$\dot q = \frac{A(I_z - I_x)pr + AI_{xz}(r^2 - p^2) - Ahr + v^2 C_m}{I_y} \tag{5}$$

$$\dot r = \frac{\left(I_x - I_y/A^3\right)pq + I_{xz}(\dot p - qr) + hq + v^2 C_n/A}{I_z} \tag{6}$$

$$\dot\delta_{l_c} = \frac{k_1\dot\alpha + k_2\alpha - \delta_{l_c}}{k_3}; \qquad \dot\delta_l = \frac{\delta_{l_c} + k_4M^2 + k_5 - \delta_l}{k_6} \tag{7}$$

$$\begin{aligned}\dot\vartheta &= q\cos\phi - \frac{r\sin\phi}{A} \\ \dot\phi &= \frac{p + (Aq\sin\phi + r\cos\phi)\tan\vartheta}{A}\end{aligned} \tag{8}$$

where the second-order leading-edge flap dynamics of the F-16 [Eqs. (7)] are included.[5] The limit Mach number for the aerodynamic model is $M = 0.6$. The aerodynamic coefficients, tabulated elsewhere,[5] in the range $-20 \le \alpha \le 90$ deg for the angle of attack and $-30 \le \beta \le 30$ deg for the sideslip angle, have the form

$$\begin{aligned}\begin{bmatrix} C_{(\mathrm{long})} \\ C_{(\mathrm{lat})} \end{bmatrix} &= \begin{bmatrix} C_{(\mathrm{long})_0}(\alpha, \beta, \delta_e, \delta_l) \\ C_{(\mathrm{lat})_0}(\alpha, \beta, \delta_e, \delta_a, \delta_r, \delta_l) \end{bmatrix} \\ &+ \frac{1}{v}\begin{bmatrix} C_{(\mathrm{long})_q}(\alpha, \delta_l)q \\ C_{(\mathrm{lat})_p}(\alpha, \delta_l)p + C_{(\mathrm{lat})_r}(\alpha, \delta_l)r \end{bmatrix}\end{aligned} \tag{9}$$

where $C_{(\mathrm{long})} = (C_x, C_z, C_m)^T$ and $C_{(\mathrm{lat})} = (C_y, C_l, C_n)^T$. We only recall here that a curve fit of the aerodynamic data, having continuous first derivative, is required for the convergence of the continuation method.

Focusing on the nonsymmetrical equilibria of the aircraft, we first search for the steady states of the reduced, eighth-order set of equations, obtained by setting the gravity equal to zero, i.e., $C_W = 0$, so as to uncouple the dynamic equations (1–6) from the Euler angle equations (8). To this end, Eq. (1), written for $\dot v = 0$, is used to determine the thrust coefficient $C_T = -(C_x + C_y\tan\beta/\cos\alpha + C_z\tan\alpha)$, which is then substituted into Eqs. (2) and (3). For constant velocity ($v = 1$) the result is

$$\dot\alpha = q - \frac{1}{A}(r\sin\alpha + p\cos\alpha)\tan\beta + \frac{C_y\sin\alpha\tan\beta + C_z}{2\mu\cos\alpha\cos\beta} \tag{10}$$

$$\dot\beta = \frac{1}{A}(p\sin\alpha - r\cos\alpha) + \frac{C_y}{2\mu\cos\beta} \tag{11}$$

We consider now the seventh-order set of governing equations (4–7), (10), and (11). When the dependence of δ_l on M in Eq. (7) is neglected because of the low subsonic speed involved in this analysis, there is only one parameter in such a set, namely h, that is a function of the aircraft state at equilibrium through the flight velocity V_e. Thus, as a further hypothesis, we neglect h in this phase to obtain a problem that is independent of equilibrium velocity. As a major advantage, the equilibrium points of the above set hold for any value of the flight speed, provided that the thrust $T = 0.5\rho V_e^2 SC_T$ assumes reasonable values. Note that, in the equilibria considered in the F-16 model, the contribution of the gyroscopic terms to the right-hand side of Eqs. (5) and (6) was at least three orders of magnitude smaller than the contribution of the aerodynamic and inertial terms. Moreover, such an assumption can be removed when the equilibrium points of the complete model are searched for, as we will see.

In the present procedure, the steady-state problem is solved for an assigned value of the control vector, i.e., $\bar{\boldsymbol{u}} = (\bar\delta_a, \bar\delta_e, \bar\delta_r)^T$. We begin by setting $\dot\alpha = \dot\beta = 0$ in Eqs. (7), (10), and (11) and by expressing the leading-edge flap angle at equilibrium as $\delta_l = k_5 + \alpha k_2$. Then, we substitute the aerodynamic coefficients C_y and C_z given by Eq. (9) into Eqs. (10) and (11) to obtain a linear set of two algebraic equations in p, q, and r, which is used to express q and r in terms of p, as follows:

$$q = Q_0(\alpha, \beta) + Q_p(\alpha, \beta)p; \qquad r = R_0(\alpha, \beta) + R_p(\alpha, \beta)p \tag{12}$$

where

$$Q_0 = CR_0 - \frac{C_{z_0}(\alpha, \beta, \bar\delta_e) + C_{y_0}(\alpha, \beta, \bar\delta_a, \bar\delta_r)\sin\alpha\tan\beta}{2\mu\cos\alpha\cos\beta + C_{z_q}(\alpha)} \tag{13}$$

$$Q_p = \frac{(2\mu/A)\cos^2\alpha\sin\beta - C_{y_p}(\alpha)\sin\alpha\tan\beta}{2\mu\cos\alpha\cos\beta + C_{z_q}(\alpha)} + CR_p \tag{14}$$

$$R_0 = \frac{C_{y_0}(\alpha, \beta, \bar\delta_a, \bar\delta_r)}{(2\mu/A)\cos\alpha\cos\beta - C_{y_r}(\alpha)} \tag{15}$$

$$R_p = \frac{(2\mu/A)\sin\alpha\cos\beta + C_{y_p}(\alpha)}{(2\mu/A)\cos\alpha\cos\beta - C_{y_r}(\alpha)} \tag{16}$$

$$C = \frac{\sin\alpha\tan\beta\left[(2\mu/A)\cos\alpha\cos\beta - C_{y_r}(\alpha)\right]}{2\mu\cos\alpha\cos\beta + C_{z_q}(\alpha)} \tag{17}$$

Substitution of q and r, as given in Eqs. (12), in Eq. (4) with $\dot{p} = 0$ and C_l as in Eq. (9), yields the following quadratic equation for p, and related coefficients:

$$L_1(\alpha,\beta)p^2 + L_2(\alpha,\beta)p + L_3(\alpha,\beta) = 0 \tag{18}$$

$$L_1 = \left(I_y/A^3 - I_z\right)Q_pR_p + I_{xz}Q_p \tag{19}$$

$$L_2 = \left(\frac{I_y}{A^3} - I_z\right)(Q_0R_p + Q_pR_0) + I_{xz}Q_0 + \frac{C_{l_p}(\alpha) + C_{l_r}(\alpha)R_p}{A} \tag{20}$$

$$L_3 = \left(\frac{I_y}{A^3} - I_z\right)Q_0R_0 + \frac{C_{l_0}(\alpha,\beta,\bar{\delta}_a,\bar{\delta}_e,\bar{\delta}_r) + C_{l_r}(\alpha)R_0}{A} \tag{21}$$

The real roots of Eq. (18), namely p_1 and p_2 for $\Delta = L_2^2 - 4L_1L_3 > 0$, are finally substituted in Eqs. (5) and (6), making use of Eqs. (12) to calculate a couple of residuals $\dot{q}$ and $\dot{r}$ for each value of roll velocity p. When this procedure is carried out for the whole range of variation of aerodynamic angles, we are able to evaluate, for each root of Eq. (18), the functions $\sigma_q(\alpha,\beta) = \dot{q}$ and $\sigma_r(\alpha,\beta) = \dot{r}$.

To compute the equilibrium points of the system, we must solve the nonlinear set

$$\sigma_q(\alpha,\beta) = 0; \qquad \sigma_r(\alpha,\beta) = 0 \tag{22}$$

in terms of angle of attack and sideslip angle. To solve this problem, first we draw in the α–β plane the zero-level contour plots of σ_q and σ_r, as shown in Fig. 1 for $p = p_1$ and $\bar{u} = 0$, i.e., no control deflection. The shaded region corresponds to values of α and β, such that $\Delta < 0$, whereas the solid and dashed lines are the zero contour levels of σ_q and σ_r, respectively. The equilibrium points of the seventh-order model are easily identified as the points of intersection of the two families of lines. In the reported case, we see that all equilibria are in the region $30 \leq \alpha \leq 50$ deg and $0 \leq \beta \leq 10$ deg and, as shown in Fig. 1, the contour lines can be redrawn at the required level of detail. Once an initial estimate of the equilibrium point has been graphically identified, then, as a second step, the true point can be determined by a Newton method algorithm. Because no solution to set (22) was found by the same analysis for the root $p = p_2$ of Eq. (18), the steady states of the F-16 zero-gravity model are those indicated in Fig. 1. We emphasize that, as far as h is neglected, the obtained solutions represent all possible equilibria of the vehicle for the assigned control vector and for any flight speed. The effect of V_e is, of course, in the thrust level (T).

The equilibrium points of the complete model are determined, as we said, by a continuation algorithm,[4] using the above solutions as starting points where, for $C_W = 0$, it is $\phi = \tan^{-1}[q/(Ar)]$ and $\vartheta = \tan^{-1}[-Ap/(q\sin\phi + Ar\cos\phi)]$. In particular, as C_W is varied, the steady states of the zero-gravity model are continued to the steady states of the set of Eqs. (4–8), (10), and (11). In so doing, once the thrust coefficient is determined at equilibrium, the thrust force is calculated as $T = mgC_T/C_W$. Also, the flight speed is $V_e = [mg/(0.5\rho SC_W)]^{1/2}$ and, finally, the effect of H can be brought into the solution by letting h vary as a function of C_W during the continuation, i.e., $h = 4H(C_W)^{1/2}/[b^2(2\rho Smg)^{1/2}]$.

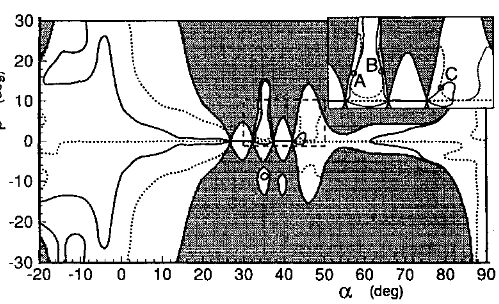

Fig. 1 Contour plot of $\sigma_q = 0$ (——) and $\sigma_r = 0$ (· · ·); $u = 0$; shaded area indicates $\Delta < 0$.

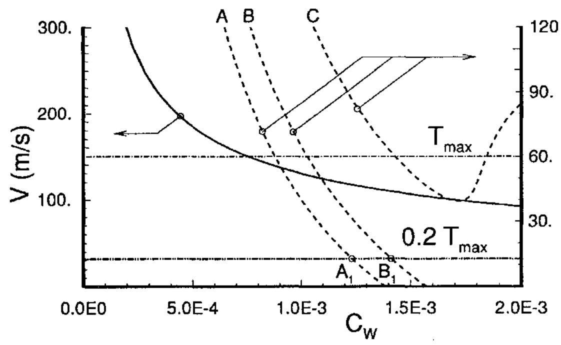

Fig. 2 Flight velocity V_e and thrust T vs C_W; the branch labels refer to the equilibrium points of the reduced-order model.

Figure 2 shows the variation of V_e and T vs C_W, where the dashed lines for T are continuations from the three equilibria of the reduced-order model indicated in Fig. 1 (labels A, B, and C). At this point we have determined the steady states of the vehicle for $\boldsymbol{u} = 0$ and a thrust value from zero to T_{max} to which a flight speed in the interval 100–150 m/s corresponds. These states are ascending or descending helical motions with vertical axis and angular velocity ω.

A final question regards a nontrivial aspect of the problem: How many states are left that are not revealed by the above technique? In this respect, the equilibrium points of the complete model were searched by the method reported by Adams,[2] for $T/T_{max} = 0.2$, using a six-dimensional grid ($v, \alpha, \beta, |\omega|, \theta, \phi$) of starting points for a nonlinear programming iterative algorithm and running as many as 6.3×10^4 computations. Even though the grid was rather coarse with respect to the six parameters, once local minima and symmetrical equilibria were disregarded, we verified that all initial states were attracted by the two solutions of Fig. 2 (points A_1 and B_1). This is a strong indication that the number of equilibrium points remains unchanged as the order of the system is decreased, at least in this application.

Conclusion

A method is proposed for the evaluation of the steady states of an aircraft when gravity is neglected and the controls are assigned. By a continuation algorithm, the equilibrium points of the complete dynamic model of the vehicle then can be computed. As a limitation, the method only applies to aircraft models where the aerodynamic coefficients depend linearly on the angular velocity components. In those cases where this condition is not met, the simple quadratic form of Eq. (18) is no longer retained.

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